VARIATIONAL METHODS FOR ALMOST PERIODIC SOLUTIONS OF A CLASS OF NEUTRAL DELAY EQUATIONS

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Abstract. We provide new variational settings to study the a.p.(almost periodic) solutions of a class of nonlinear neutral delay equations. We extend a Shu and Xu's [22] variational setting for periodic solutions of nonlinear neutral delay equation to the almost periodic settings. We obtain results on the structure of the set of the a.p. solutions, results of existence of a.p. solutions, results of existence of a.p. solutions, and also a density result for the forced equations.

1. Introduction

The aim of this paper is the study of a.p. (almost periodic) solutions of neutral delay equations of the following form :

$$\begin{cases} D_1L(x(t-r), x(t-2r), x'(t-r), x'(t-2r), t-r) \\ +D_2L(x(t), x(t-r), x'(t), x'(t-r), t) \\ = \frac{d}{dt}[D_3L(x(t-r), x(t-2r), x'(t-r), x'(t-2r), t-r) \\ +D_4L(x(t), x(t-r), x'(t), x'(t-r), t)] \end{cases} \tag{1.1}$$

where $L : (\mathbb{R}^n)^4 \times \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function; D_j denotes the partial differential with respect to the j^{th} vector variable, and $r \in (0, \infty)$ is fixed. We will consider the almost periodicity in the sense of Bohr [14], and in the sense of Besicovitch [2].

A special case of (1.1) is the following forced neutral delay equation.

$$\begin{cases} D_1K(x(t-r), x(t-2r), x'(t-r), x'(t-2r)) \\ +D_2K(x(t), x(t-r), x'(t), x'(t-r)) \\ -\frac{d}{dt}[D_3K(x(t-r), x(t-2r), x'(t-r), x'(t-2r)) \\ +D_4K(x(t), x(t-r), x'(t), x'(t-r))] = b(t) \end{cases} \tag{1.2}$$

where $K : (\mathbb{R}^n)^4 \to \mathbb{R}$ is a differentiable function, and $b : \mathbb{R} \to \mathbb{R}^n$ is an a.p. forcing term. To see (1.2) as a special case of (1.1) it suffices to take

$$L(x_1, x_2, x_3, x_4, t) := K(x_1, x_2, x_3, x_4) - x_1.b(t+r)$$

where the point denotes the usual inner product in $\mathbb{R}^n$.

Another special case of (1.1) is the following forced second-order neutral delay equation :

$$x''(t-r) + D_1F(x(t-r), x(t-2r)) + D_2F(x(t), x(t-r)) = b(t)$$

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Where $b : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $F : (\mathbb{R}^n)^2 \longrightarrow \mathbb{R}$. To see that this last equation is a special case of (1.1) it suffices to take $L(x_1, x_2, x_3, x_4, t) := \frac{1}{2} \|x_3\|^2 - F(x_1, x_2) + x_1.b(t+r)$, where the norm is the usual Euclidian norm of $\mathbb{R}^n$. In their work [22], Shu and Xu study the periodic solutions of this last equation by using a variational method. We want to extend such a view point to the study of the a.p. solutions.

And so our approach to the study of the a.p. solutions of (1.1) consists to search critical points of a functional Φ defined on suitable Banach spaces of a.p functions by :

$$\Phi(x) := \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} L(x(t), x(t-r), x'(t), x'(t-r), t)dt \tag{1.3}$$

At this time we give some historical elements. Recall that the work [16] of Elsgolc treats the calculus of variations with a retarded argument on a bounded real interval. This work was followed by these ones of Hughes [17] and Sabbagh [20]. Since the variational problems can be seen as optimal control problems, recall also the existence of the theory of the Periodic Optimal Control with retarded argument as developped by Colonius in [13]. For instance, we consider a periodic Optimal Control problem with a criterion of the form $\frac{1}{T}\int_0^T g(x(t), u(t), t)dt$ and with an equation of motion of the form $x'(t) = f(x(t), x(t-r), u(t), t)$, where $x(t)$ is the state variable and $u(t)$ the control variable. In the special case where $f(x(t), x(t-r), u(t), t) = f_1(x(t), x(t-r), t) + u(t)$, the previous Optimal Control problem can be transformed into a calculus of variations problem with the criterion $\frac{1}{T}\int_0^T g(x(t), f_1(x(t), x(t-r), t) - x'(t), t)dt$, which is a special case of (1.3). Note that the Euler-Lagrange equation of such a variational problem is a special case of (1.1).

On another hand, Calculus of Variations in Mean Time was developped to study the a.p. solutions of some (non retarded) differential equations [3, 4, 5, 6, 7, 8]. Here, we extend this approach to treat equation like (1.1).

Now we describe the contents of this paper. In Section 2, we precise the notations about the function spaces used later. In Section 3 we etablish a variational formalism suitable to the Bohr-ap solutions; we give a variational principle and a result on the structure of the set of the a.p. solutions of (1.1) in the convex case. In Section 4, we etablish a variational formalism suitable to the Besicovitch-ap solutions, we give a variational principle, results of existence, and a result of density for the almost periodically forced equations.

2. Notations

$AP^0(\mathbb{R}^n)$ is the space of the Bohr almost periodic (Bohr-ap for short) functions from $\mathbb{R}$ in $\mathbb{R}^n$; endowed with the supremum $\|.\|_\infty$, it is a Banach space [14].
$AP^1(\mathbb{R}^n) := \left\{x \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n) \cap AP^0(\mathbb{R}^n) : x' \in AP^0(\mathbb{R}^n)\right\}$; endowed with the norm $\|x\|_{\mathcal{C}^1} := \|x\|_\infty + \|x'\|_\infty$, it is a Banach space.
When $k \in \mathbb{N}^* \cup \{\infty\}$, $AP^k(\mathbb{R}^n) := \left\{x \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^n) : \forall j \leq k, \frac{d^j x}{dt^j} \in AP^0(\mathbb{R}^n)\right\}$

When $x \in AP^0(\mathbb{R}^n)$, its mean value

$$\mathcal{M}\{x(t)\}_t := \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt \text{ exists in } \mathbb{R}^n.$$

The Fourier-Bohr coefficients of $x \in AP^0(\mathbb{R}^n)$ are the complex vectors

$$a(x;\lambda) := \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}x(t)dt$$

and $\Lambda(x) := \{\lambda \in \mathbb{R} : a(x,\lambda) \neq 0\}$.

When $p \in [1,\infty)$, $B^p(\mathbb{R}^n)$ is the completion of $AP^0(\mathbb{R}^n)$ (in $L^p_{loc}(\mathbb{R},\mathbb{R}^n)$) with respect to the norm $\|u\|_p := \mathcal{M}\{|u|^p\}^{\frac{1}{p}}$. When $p = 2$, $B^2(\mathbb{R}^n)$ is a Hilbert spaces and its norm $\|.\|_2$ is associated to the inner product $(u \mid v) := \mathcal{M}\{u.v\}$ [2]. The elements of these spaces $B^p(\mathbb{R}^n)$ are called Besicovitch almost periodic (Besicovitch-ap for short) functions.

Recall the useful following fact : if $(u_m)_m$ is a sequence in $AP^0(\mathbb{R}^n)$ and if $u \in L^p_{loc}(\mathbb{R},\mathbb{R}^n)$ (Lebesgue space), which satisfy

$$\overline{\mathcal{M}}\{\|u_m - u\|^p\}^{\frac{1}{p}} = \left(\limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|u_m - u\|^p\, dt\right)^{\frac{1}{p}} \to 0 \quad (m \to 0)$$

then $u \in B^p(\mathbb{R}^n)$ and we have $\|u_m - u\|_p \to 0$ $(m \to 0)$.

We use the generalized derivative $\nabla u \in B^2(\mathbb{R}^n)$ of $u \in B^2(\mathbb{R}^n)$ (when it exists) defined by $\left\|\nabla u - \frac{1}{s}(u(.+s) - u)\right\|_2 \to 0$ $(s \to 0)$, and we define $B^{1,2}(\mathbb{R}^n) := \left\{u \in B^2(\mathbb{R}^n) : \nabla u \in B^2(\mathbb{R}^n)\right\}$; endowed with the inner product $\langle u \mid v\rangle := (u \mid v) + (\nabla u \mid \nabla v)$, $B^{1,2}(\mathbb{R}^n)$ is a Hilbert space, [6, 8].

If E and F are two finite-dimentional normed spaces, $APU(E \times \mathbb{R}, F)$ stands for the space of the functions $f : E \times \mathbb{R} \longrightarrow F$, $(x,t) \longmapsto f(x,t)$, which are almost periodic in t uniformly with respect to x in the classical sense given in [23].

To make the writing less heavy, we sometimes use the notations

$$\underline{u}(t) := (u(t), u(t-r), \nabla u(t), \nabla u(t-r))$$

when $u \in B^{1,2}(\mathbb{R}^n)$, and

$$\underline{x}(t) := (x(t), x(t-r), x'(t), x'(t-r))$$

when $x \in AP^1(\mathbb{R}^n)$.

3. A variational setting for the Bohr-ap functions

We consider the following condition :

$$\begin{cases} L \in APU((\mathbb{R}^n)^4 \times \mathbb{R}, \mathbb{R}), \text{ and, for all } (X,t) \in (\mathbb{R}^n)^4 \times \mathbb{R}, \\ \text{the partial differential } D_X L(X,t) \text{ exists, and} \\ D_X L \in APU((\mathbb{R}^n)^4 \times \mathbb{R}, \mathcal{L}((\mathbb{R}^n)^4, \mathbb{R})). \end{cases} \tag{3.1}$$

Lemma 3.1. *Under* [4] *the functional* $\Phi : AP^1(\mathbb{R}^n) \longrightarrow \mathbb{R}$ *defined by* (1.3) *is of class* $\mathcal{C}^1$*, and for all* $x, h \in AP^1(\mathbb{R}^n)$ *we have*

$$\begin{aligned} D\Phi(x).h \quad = \quad & \mathcal{M}\{D_1L(\underline{x}(t),t).h(t) + D_2L(\underline{x}(t),t).h(t-r) \\ & + D_3L(\underline{x}(t),t).h'(t) + D_4L(\underline{x}(t),t).h'(t-r)\}_t. \end{aligned}$$

Proof. We introduce the linear operator $\mathcal{T} : AP^1(\mathbb{R}^n) \longrightarrow (AP^0(\mathbb{R}^n))^4$ by setting $\mathcal{T}(x)(t) := \underline{x}(t)$. The four components of $\mathcal{T}$ are continuous linear operators that implies the continuity of $\mathcal{T}$, and therefore $\mathcal{T}$ is of class $\mathcal{C}^1$, and for all $x, h \in AP^1(\mathbb{R}^n)$ we have $D\mathcal{T}(x).h = \mathcal{T}(h)$.

Under (3.1), the Nemytski operator $\mathcal{N}_L : (AP^0(\mathbb{R}^n))^4 \longrightarrow AP^0(\mathbb{R}^n)$, defined by $\mathcal{N}_L(X)(t) := L(X(t), t)$, is of class $\mathcal{C}^1$, cf. Lemma 7 in [9], and we have, for all $X, H \in AP^0(\mathbb{R}^n)^4$, $(D\mathcal{N}_L(X).H)(t) = D_X L(X(t), t).H(t)$.

The linear functional $\mathcal{M} : AP^0(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is continuous, therefore it is of class $\mathcal{C}^1$ and we have, for all $\phi, \psi \in AP^0(\mathbb{R}^n)$, $D\mathcal{M}\{\phi\}.\psi = \mathcal{M}\{\psi\}$.

And so $\Phi = \mathcal{M} \circ \mathcal{N}_L \circ \mathcal{T}$ is of class $\mathcal{C}^1$. Furthermore, we have

$$\begin{aligned} D\Phi(x).h &= D\mathcal{M}(\mathcal{N}_L \circ \mathcal{T}(x)) \circ D\mathcal{N}_L(\mathcal{T}(x)) \circ D\mathcal{T}(x).h \\ &= \mathcal{M}\{D\mathcal{N}_L(\mathcal{T}(x)).\mathcal{T}(h)\} \\ &= \mathcal{M}\{D_X L(\underline{x}(t), t).\underline{h}(t)\}_t \end{aligned}$$

and expressing $D_X L$ in terms of $D_j L$ we obtain the annonced formula. □

Note that in the case without delay, when L is autonomous, i.e. $L(X, t) = L(X)$, in [4] it is established that the functional $x \longmapsto \mathcal{M}\{L(x, x')\}$ is of class $\mathcal{C}^1$ when L is of class $\mathcal{C}^1$. In [12] we can find a proof of the differentiability of the Nemytskii operator on $AP^0(\mathbb{R}^n)$ which is different to this one of [4].

Theorem 3.2 (Variational Principle)**.** *Under* (3.1)*, for $x \in AP^1(\mathbb{R}^n)$, the two following assertions are equivalent.*

(i) *$D\Phi(x) = 0$, i.e. x is a critical point of Φ in $AP^1(\mathbb{R}^n)$.*

(ii) *x is a Bohr-ap solution of the equation* (1.1)*.*

Proof. First we assume (i). Since the mean value is translation invariant, we have

$$\mathcal{M}\{D_2 L(\underline{x}(t), t).h(t-r)\}_t = \mathcal{M}\{D_2 L(\underline{x}(t+r), t+r).h(t)\}_t$$

and

$$\mathcal{M}\{D_4 L(\underline{x}(t), t).h'(t-r)\}_t = \mathcal{M}\{D_4 L(\underline{x}(t+r), t+r).h'(t)\}_t,$$

and so by using Lemma 3.1 we obtain, for all $h \in AP^1(\mathbb{R}^n)$,

$$\begin{aligned} 0 &= \mathcal{M}\{(D_1 L(\underline{x}(t), t) + D_2 L(\underline{x}(t+r), t+r)).h(t)\}_t \\ &+ \mathcal{M}\{(D_3 L(\underline{x}(t), t) + D_4 L(\underline{x}(t+r), t+r)).h'(t)\}_t. \end{aligned}$$

Setting $q(t) := D_1 L(\underline{x}(t), t) + D_2 L(\underline{x}(t+r), t+r)$, denoting by $q_k(t)$ its coordinates for $k = 1, \ldots, n$, setting $p(t) := D_3 L(\underline{x}(t), t) + D_4 L(\underline{x}(t+r), t+r)$, and denoting by $q_k(t)$ its coordinates for $k = 1, \ldots, n$, we deduce from the previous equality that, for all $\phi \in AP^\infty(\mathbb{R})$ we have $\mathcal{M}\{q_k(t).\phi(t)\}_t = -\mathcal{M}\{p_k(t).\phi'(t)\}_t$. Then by reasoning like in the proof of Theorem 1 in [3], we obtain that $Dp_k = q_k$ in the sens of the ap distributions of Schwartz [21], and by using the proposition of the Fourier-Bohr series we obtain that p_k is $\mathbb{C}^1$ and that $p'_k = q_k$ in the ordinary sense. From this, we obtain that $p(.-r)$ is $\mathbb{C}^1$ and that $p'(t-r) = q(t-r)$ which is exactly (ii).

Conversely by using he formula $\mathcal{M}\{l.y'\} = -\mathcal{M}\{l'.y\}$ for all $l \in AP^1(\mathcal{L}(\mathbb{R}^n, \mathbb{R}))$ and $y \in AP^1(\mathbb{R}^n)$, and by translating the time, we obtain from (ii) for all $h \in AP^1(\mathbb{R}^n)$ the following relation

$$\begin{aligned} 0 &= \mathcal{M}\{(D_1 L(\underline{x}(t), t) + D_2 L(\underline{x}(t+r), t+r)).h(t) \\ &\quad +(D_3 L(\underline{x}(t), t) + D_4 L(\underline{x}(t+r), t+r)).h'(t)\}_t \\ &= \mathcal{M}\{D_1 L(\underline{x}(t), t)h(t) + D_2 L(\underline{x}(t), t)).h(t-r) \\ &\quad +D_3 L(\underline{x}(t), t)h'(t) + D_4 L(\underline{x}(t), t).h'(t-r)\}_t \\ &= D\Phi(x).h, \end{aligned}$$

and so we have (i). $\square$

This Theorem 3.2 is an extension to the non autonomous case in presence of a delay of Theorem 1 in [3]. Now we use Theorem 3.2 to provide some results on the structure of the set of the Bohr-ap solutions of (1.1) in the case where L is autonomous and convex.

Theorem 3.3 (Structure Result). *We assume that $L \in \mathcal{C}^1((\mathbb{R}^n)^4, \mathbb{R})$, and that L is convex. Then the following assertation hold.*

(i) *The set of the Bohr-ap solutions of* (1.1) *is a closed convex subset of $AP^1(\mathbb{R}^n)$.*
(ii) *If x^1 is a T^1-periodic non constant solution of* (1.1)*, if x^2 is a T^2-periodic non constant solution of* (1.1)*, and if T^1/T^2 is no rational, then $(1-\theta)x^1 + \theta x^2$ is a Bohr-ap non periodic solution of* (1.1) *for all $\theta \in (0, 1)$.*
(iii) *If x is a Bohr-ap solution of* (1.1)*, then $\mathcal{M}\{x\}$ is a constant solution of* (1.1)*.*
(iv) *If x is a Bohr-ap solution of* (1.1)*, if $T \in (0, \infty)$ is such that $a(x, \frac{2\pi}{T}) \neq 0$, then there exists a non constant T-periodic solution of* (1.1)*.*

Proof. Since L is convex, the functional Φ of (1.3) is also convex on $AP^1(\mathbb{R}^n)$. Since L is autonomous and of class $\mathcal{C}^1$, L satisfies (3.1), and so Φ is of class $\mathcal{C}^1$. Therefore we have $\{x : \Phi(x) = \inf \Phi\} = \{x : D\Phi(x) = 0\}$ which is closed and convex, and (i) becomes a consequence of Theorem 3.2. The assertation (ii) is a straightfoward consecónce of (i).

We introduce $C_{T,\nu}(x)(t) := \frac{1}{\nu}\sum_{k=0}^{\nu-1} x(t + kT)$, when x is a Bohr-ap solution of (1.1), for all $\nu \in \mathbb{N}^*$. By using a Theorem of Besicovitch, [2] p.144, there exists a T-periodic continuous function, denoted by x^T, such that $\lim_{\nu\to\infty} \left\|C_{T,\nu}(x) - x^T\right\|_\infty = 0$.

We easily verify that $\lim_{\nu\to\infty} \left\|C_{T,\nu}(x) - x^T\right\|_{\mathbb{C}^1} = 0$.

Since L is autonomous, $t \mapsto x(t + kT)$ is a Bohr-ap solution of (1.1). Since $C_{T,\nu}(x)$ is a convex combinaison of Bohr-ap solutions of (1.1), $C_{T,\nu}(x)$ is a Bohr-ap solution of (1.1), and x^T also by using the closedness of the set of Bohr-ap solutions. And so x^T is a T-periodic solution of (1.1). By using a straightforward calculation we see that $a(C_{T,\nu}(x), \frac{2\pi}{T}) = a(x, \frac{2\pi}{T})$ and consequently $a(x^T, \frac{2\pi}{T}) = a(x, \frac{2\pi}{T})$. When $a(x, \frac{2\pi}{T}) \neq 0$ then x^T is not constant that proves (iv).

To prove (iii) it suffices to choose $T^1 \in (0, \infty)$ such that $\frac{2\pi}{T^1}(\mathbb{Z} - \{0\}) \cap \Lambda(x) = 0$, and then all the Fourier-Bohr coefficients of x^{T^1} are zero except (perhaps) the mean value of x^{T^1} which is equal to $\mathcal{M}\{x\}$. $\square$

The assertions (i) and (ii) are extensions of the Theorems 3 and 4 in [3]; the assertions (iii) et (iv) are extension to neutral delay equations of Theorem 2 in [5].

The space $(AP^1(\mathbb{R}^n), \|.\|_{\mathbb{C}^1})$ does not possess good topological properties like to be a reflexive space. It is why in the following section we extend our variational formalism to the Hilbert space $B^{1,2}(\mathbb{R}^n)$.

4. A variational setting for the Besicovitch-ap functions

E and F are Euclidean finite-dimensional spaces.

Lemma 4.1. *Let $g \in APU(E \times \mathbb{R}, F)$ be a function which satisfies the following Hölder condition :*

$$\exists \alpha \in (0,\infty), \exists a \in [0,\infty), \forall t \in \mathbb{R}, \forall z, w \in E, \quad |g(z,t) - g(w,t)| \leq a. |z-w|^{\alpha}$$

Let $p, q \in [1,\infty)$ be such that $p = \alpha q$.
Then the two following assertions hold.

(i) *If $u \in B^p(E)$ then $t \mapsto g(u(t),t) \in B^q(F)$.*
(ii) *The Nemytskii operator on g, $\mathcal{N}_g : B^p(E) \to B^q(F)$ defined by $\mathcal{N}_g u(t) := g(u(t),t)$, satisfies $\|\mathcal{N}_g u - \mathcal{N}_g v\|_q \leq a. \|u-v\|_p^{\alpha}$ for all $u, v \in B^p(E)$.*

Proof. We set $b(t) := g(0,t)$, and so we have $b \in AP^0(\mathbb{R})$ and the Hölder assumption implies $|g(x,t)| \leq a. |z|^{\alpha} + b(t)$ for all $z \in E, t \in \mathbb{R}$. If $u \in B^p(E)$ then we have $|g(u(t),t)| \leq a. |u(t)|^{\alpha} + b(t)$ for all $t \in \mathbb{R}$, and since b is continuous we have $b \in L^q_{loc}(\mathbb{R},\mathbb{R})$ (the Lebesgue space), and since $(|u(t)|^{\alpha})^q = |u(t)|^p$ we have $|u|^{\alpha} \in L^q_{loc}(\mathbb{R},\mathbb{R})$ Since $u \in B^p(E)$ there exists a sequence $(u_j)_j$ in $AP^0(E)$ such that $\lim_{j\to\infty} \|u - u_j\|_p = 0$. By using Theorem 2.7 in [23] p. 16, setting $\varphi_j(t) := g(u_j(t),t)$, we have $\varphi_j \in AP^0(F)$, and a straightforward calculation gives us the following inequality :

$$\overline{\mathcal{M}}\{|g(u(t),t) - \varphi_j(t)|^q\}^{\frac{1}{q}} \leq a.\mathcal{M}\{|u-u_j|^p\}^{\frac{1}{q}} = a \|u-u_j\|_p^{\alpha},$$

and consequently we obtain

$$\lim_{j\to\infty} \overline{\mathcal{M}}\{|g(u(t),t) - \varphi_j(t)|^p\}^{\frac{1}{q}} = 0$$

that implies : $t \mapsto g(u(t),t) \in B^q(F)$, and so (i) is proven; moreover the last previous inequality becomes this one of (ii) when we replace $\varphi_j(t)$ by $g(v(t),t)$. □

This lemma is an extension to the non autonomous case of Theorem 1 in [8].

Lemma 4.2. *Let $f \in APU(E\times\mathbb{R}, F)$ be a function such that the partial differential $D_1 f(z,t)$ exists for all $(z,t) \in E \times \mathbb{R}$ and such that $D_1 f \in APU(E \times \mathbb{R}, \mathcal{L}(E,F))$. We assume the following condition fulfilled.*

(C) *There exist $a_1 \in [0,\infty)$, such that, for all $z, w \in E$, and for all $t \in \mathbb{R}$, $|D_1 f(z,t) - D_1 f(w,t)| \leq a_1. |z-w|$.*

Then the Nemytskii operator $\mathcal{N}_f : B^2(E) \to B^1(F)$, defined by $\mathcal{N}_f(u)(t) := f(u(t),t)$, is of class $\mathcal{C}^1$ and, for all $u, h \in B^2(E)$ we have

$$(D\mathcal{N}_f(u).h)(t) = D_1 f(u(t),t).h(t).$$

Proof. First step : We show that there exist $a_0 \in [0,\infty)$, $b \in B^1(E)$, such that, for all $(z,t) \in E \times \mathbb{R}$, $|f(z,t)| \leq a_0 |z|^2 + b(t)$.
$|D_1 f(z,t) - D_1 f(0,t)| \leq a_1. |z|$

$$\begin{aligned} \Rightarrow |D_1 f(z,t)| &\leq |D_1 f(z,t) - D_1 f(0,t)| + |D_1 f(0,t)| \\ &\leq a_1. |z| + |D_1 f(0,t)|. \end{aligned}$$

By using the mean value theorem, [1] p. 144, we have for all $(z,t) \in E \times \mathbb{R}$,

$$\begin{aligned}
|f(z,t)| &\leq |f(z,t) - f(0,t)| + |f(0,t)| \\
&\leq \sup_{\xi \in]0,z[} |D_1 f(\xi,t)| \,.\, |z - 0| + |f(0,t)| \\
&\leq \sup_{\xi \in]0,z[} (a_1 .\, |\xi| + |D_1 f(0,t)|) .\, |z| + |f(0,t)| \\
&= (a_1 .\, |z| + |D_1 f(0,t)|) .\, |z| + |f(0,t)| \\
&= a_1 .\, |z|^2 + |D_1 f(0,t)| \,.\, |z| + |f(0,t)| \\
&\leq a_1 .\, |z|^2 + \frac{1}{2} |D_1 f(0,t)|^2 + \frac{1}{2} |z|^2 + |f(0,t)| \\
&= (a_1 + \frac{1}{2}) .\, |z|^2 + \frac{1}{2} |D_1 f(0,t)|^2 + |f(0,t)| \,.
\end{aligned}$$

Setting $b(t) := \frac{1}{2} |D_1 f(0,t)|^2 + |f(0,t)|$, and $a_0 := a_1 + \frac{1}{2}$. Since $f \in APU(E \times \mathbb{R}, F)$, and $D_1 f \in APU(E \times \mathbb{R}, \mathcal{L}(E,F))$, we have $b \in AP^0(E) \subset B^1(E)$.

Second step : We show that $t \mapsto f(u(t),t) \in B^1(F)$ when $u \in B^2(E)$.
Let $u \in B^2(E)$. Then the inequality $|f(u(t),t)| \leq a_0 |u(t)|^2 + b(t)$ implies that

$$t \mapsto f(u(t),t) \in L^1_{loc}(\mathbb{R}, F).$$

By using Lemma 4.1 with $p = 2, q = 2, \alpha = 1$, and $g = D_1 f$ we have $t \mapsto D_1 f(u(t),t) \in B^2(\mathcal{L}(E,F))$. Let $(u_m)_m$ be a sequence in $AP^0(E)$ such that $\|u - u_m\|_2 \to 0$ $(m \to \infty)$. By using the mean value theorem, [1] p. 144, we have, for all $t \in \mathbb{R}$,

$$\begin{aligned}
&|f(u_m(t),t) - f(u(t),t) - D_1 f(u(t),t).(u_m(t) - u(t))| \\
&\leq (\sup_{\xi \in]u(t),u_m(t)[} |D_1 f(\xi,t) - D_1 f(u(t),t)|) .\, |(u_m(t) - u(t))| \\
&\leq a_1 .\, \sup_{\xi \in]u(t),u_m(t)[} |\xi - u(t)| \,.\, |u_m(t) - u(t)| \leq a_1 .\, |u_m(t) - u(t)|^2 \,,
\end{aligned}$$

and consequently we obtain

$$\overline{\mathcal{M}}\{|f(u_m(t),t) - f(u(t),t) - D_1 f(u(t),t).(u_m(t) - u(t))|\}_t \leq a_1 .\, \|u_m - u\|_2^2$$

Since $t \mapsto D_1 f(u(t),t) \in B^2(\mathcal{L}(E,F))$ and since $u_m - u \in B^2(E)$, we have

$$t \mapsto D_1 f(u(t),t).(u_m(t) - u(t)) \in B^1(F).$$

By using Theorem 2.7 in [23] p. 16, we have

$$t \mapsto f(u_m(t),t) \in AP^0(F) \subset B^1(F),$$

and so, by setting

$$\psi_m(t) := f(u_m(t),t) - D_1 f(u_m(t),t).(u_m(t) - u(t))$$

we have $\psi_m \in B^1(F)$. The last previous inequality implies

$$\lim_{n \to \infty} \overline{\mathcal{M}}\{|f(u(t),t) - \psi_m(t)|\}_t = 0,$$

and therefore we have $t \mapsto f\left(u\left(t\right), t\right) \in B^1\left(F\right)$.

Third step : We show that, for all $u \in B^2(E)$, the opertor $\mathcal{L}(u) : B^2 \to B^1(\mathbb{R})$, defined by $(\mathcal{L}(u).h)(t) := D_1 f(u(t),t).h(t)$, is linear continuous. We have yet seen that $t \mapsto D_1 f(u(t),t).h(t) \in B^1(F)$. The linearity of $\mathcal{L}(u)$ is easy to verify. By using a Cauchy-Schwartz-Buniakovski inequality we have

$$\begin{aligned} \mathcal{M}\{|D_1 f(u(t),t).h(t)|\}_t &\leq \mathcal{M}\{|D_1 f(u(t),t)| \,.\, |h(t)|\}_t \\ &\leq \mathcal{M}\{|D_1 f(u(t),t)|^2\}^{\frac{1}{2}}.\mathcal{M}\{|h|^2\}^{\frac{1}{2}} \end{aligned}$$

that proves the continuity of $\mathcal{L}(u)$.

Fourth step : We show the differentiability of $\mathcal{N}_f$.

Let $u \in B^2(E)$ and $h \in B^2(E)$. By using the mean value inequality, [1] p. 144, we have for all $t \in \mathbb{R}$,

$$\begin{aligned} & |f(u(t)+h(t),t) - f(u(t),t) - D_1 f(u(t),t).h(t)| \\ \leq & \sup_{\xi \in]u(t),u(t)+h(t)[} |D_1 f(\xi,t) - D_1 f(u(t),t)| \,.\, |h(t)| \leq a_1 \|h\|^2 , \end{aligned}$$

and by using the monotonicity of $\mathcal{M}$ we obtain

$$\mathcal{M}\{|f(u(t)+h(t),t) - f(u(t),t) - D_1 f(u(t),t).h(t)|\}_t \leq a_1 \|h\|_2^2 ,$$

i.e.

$$\|\mathcal{N}_f(u+h) - \mathcal{N}_f(u) - \mathcal{L}(u).h\|_1 \leq a_1 \|h\|_2^2$$

that implies that $\mathcal{N}_f$ is differentiable at u and that $D\mathcal{N}_f(u) = \mathcal{L}(u)$.

Fifth step : We show that $\mathcal{N}_f$ is of class $\mathcal{C}^1$.

Let $u, v \in B^2(E)$. By using (C), for all $h \in B^2(E)$, such that $\|h\|_2 \leq 1$, for all $t \in \mathbb{R}$ we have :

$$\begin{aligned} |(D_1 f(u(t),t) - D_1 f(v(t),t)).h(t)| &\leq |D_1 f(u(t),t) - D_1 f(v(t),t)| \,.\, |h(t)| \\ &\leq a_1 . \,|u(t) - v(t)| \,.\, |h(t)| \end{aligned}$$

That implies, by using the Cauchy-Schwartz-Buniakovski inequality, the following majorization :

$$\begin{aligned} & \mathcal{M}\{|(D_1 f(u(t),t) - D_1 f(v(t),t)).h(t)|\}_t \\ \leq & a_1 \mathcal{M}\{|u(t)-v(t)| \,.\, |h(t)|\}_t \leq a_1 \|u-v\|_2 \,.\, \|h\|_2 \leq a_1 \|u-v\|_2 . \end{aligned}$$

Therefore we have $\|D\mathcal{N}_f(u) - D\mathcal{N}_f(v)\|_{\mathcal{L}} \leq a_1 \|u-v\|_2$ that implies the continuity of $D\mathcal{N}_f$. □

Note that Lemma is an extension to the non autonomous case of Theorem 2 in [8].

Theorem 4.3 (Variational Principle)**.** *Let* $L : (\mathbb{R}^n)^4 \times \mathbb{R} \to \mathbb{R}$,

$$(X,t) = (x_1, x_1, x_1, x_1, t) \mapsto L(X,t) = L(x_1, x_1, x_1, x_1, t),$$

be a function and let $r \in (0,\infty)$. *We assume the following conditions fulfilled.*

$$\begin{cases} L \in APU((\mathbb{R}^n)^4 \times \mathbb{R}, \mathbb{R}), \textit{the partial differentials } D_k L(x_1, x_1, x_1, x_1, t) \\ \textit{exist for all } (x_1, x_1, x_1, x_1, t) \in (\mathbb{R}^n)^4 \times \mathbb{R} \\ \textit{and for } k = 1, \dots, 4, \textit{ and } D_k L \in APU((\mathbb{R}^n)^4 \times \mathbb{R}, \mathcal{L}(\mathbb{R}^n, \mathbb{R})). \end{cases} \tag{4.1}$$

$$\begin{cases} \textit{There exists } a_1 \in [0,\infty) \textit{ such that } |L_X(X,t) - L_X(Y,t)| \leq a_1 |X - Y| \\ \textit{for all } X, Y \in (\mathbb{R}^n)^4 \textit{ and for all } t \in \mathbb{R} \textit{ where } L_X \textit{ is the partial differential} \\ \textit{with respect to } X \in (\mathbb{R}^n)^4 \end{cases} \tag{4.2}$$

Then the functional $J : B^{1,2}(\mathbb{R}^n) \to \mathbb{R}$*, defined by*

$$J(u) = \mathcal{M}\{L(u(t), u(t-r), \nabla u(t), \nabla u(t-r), t)\}_t$$

is of class $\mathcal{C}^1$*, and the two following assertions are equivalent.*

(i) $DJ(u) = 0$*, i.e.* u *is a critical point of* J*.*

(ii) $D_1L(u(t-r), u(t-2r), \nabla u(t-r), \nabla u(t-2r), t-r)$
$+D_2L(u(t), u(t-r), \nabla u(t), \nabla u(t-r), t)$
$= \nabla[D_3L(u(t-r), u(t-2r), \nabla u(t-r), \nabla u(t-2r), t-r)$
$+D_4L(u(t), u(t-r), \nabla u(t), \nabla u(t-r), t)]$
(equality in $B^2(\mathcal{L}(\mathbb{R}^n, \mathbb{R}))$*).*

Definition 4.4. *When* $u \in B^{1,2}(\mathbb{R}^n)$ *satisfies the equation of (ii) in Theorem 4.3, we say that* u *is a weak Besicovitch-ap solution of* (1.1)

Proof. We consider the operator $\mathcal{L} : B^{1,2}(\mathbb{R}^n) \to B^2(\mathbb{R}^n)^4 \equiv B^2((\mathbb{R}^n)^4)$, defined by $(\mathcal{L}(u))(t) := (u(t), u(t-r), \nabla u(t), \nabla u(t-r))$. $\mathcal{L}$ is clearly linear continuous, therofre $\mathcal{L}$ is of class $\mathcal{C}^1$ and we have $D\mathcal{L}(u).h = \mathcal{L}(h)$.

We consider the Nemytskii operator

$$\mathcal{N}_L : B^2((\mathbb{R}^n)^4) \to B^1(R), \quad (\mathcal{N}_L(u))(t) := L(u(t), t).$$

By using Lemma 4.2, $\mathcal{N}_L$ is of class $\mathcal{C}^1$ and, for all $U, H \in B^{1,2}((\mathbb{R}^n)^4)$ we have

$$\begin{aligned} (D\mathcal{N}_L(U).H)(t) &= L_X((U(t), t).H(t) \\ &= \sum_{k=1}^{4} D_kL(u_1(t), u_2(t), u_3(t), u_4(t), t).h_k(t). \end{aligned}$$

The mean value $\mathcal{M} : B^1(\mathbb{R}) \to \mathbb{R}$ is linear continuous, therefore it is of class $\mathcal{C}^1$, and $D\mathcal{M}\{\phi\}.\psi = \mathcal{M}\{\psi\}$ for all $\phi, \psi \in B^1(\mathbb{R})$.
Consequently $J = \mathcal{M} \circ \mathcal{N}_L \circ \mathcal{L}$ is of class $\mathcal{C}^1$ as a composition of three mappings of class $\mathcal{C}^1$.
Let $u \in B^{1,2}(\mathbb{R}^n)$. If (i) is true then, for all $h \in B^{1,2}(\mathbb{R}^n)$, we have

$$\begin{aligned} 0 &= DJ(u).h = D\mathcal{M}\{\mathcal{N}_L \circ \mathcal{L}(u)\} \circ D\mathcal{N}_L(\mathcal{L}(u)) \circ D\mathcal{L}(u).h \\ &= \mathcal{M}\{D\mathcal{N}_L(\mathcal{L}(u)).\mathcal{L}(h)\} \\ &= \mathcal{M}\{D_1L(\underline{u}(t), t).h(t) + D_2L(\underline{u}(t), t).h(t-r) \\ &\quad +D_3L(\underline{u}(t), t).\nabla h(t) + D_4L(\underline{u}(t), t).\nabla h(t-r)\}_t \\ &= \mathcal{M}\{(D_1L(\underline{u}(t), t) + D_2L(\underline{u}(t+r), t+r)).h(t)\}_t \\ &\quad +\mathcal{M}\{(D_3L(\underline{u}(t), t) + D_4L(\underline{u}(t+r), t+r)).\nabla h(t)\}_t \end{aligned}$$

and then we obtain (ii) by using Proposition 10 in [8].

Conversely, if (ii) is true then $t \mapsto D_3\mathcal{N}_L(\underline{u}(t), t) + D_4\mathcal{N}_L(\underline{u}(t+r), t+r) \in B^{1,2}(\mathbb{R}^n)$, and for all $h \in AP^1(\mathbb{R}^n)$ we have :

$$\mathcal{M}\{(D_1L(\underline{u}(t), t) + D_2L(\underline{u}(t+r), t+r)).h(t)\}_t$$

$$-\mathcal{M}\{\nabla(D_3L(\underline{u}(t), t) + D_4L(\underline{u}(t+r), t+r)).h(t)\}_t = 0$$

therefore by using Proposition 9 in [8] we obtain :

$$\begin{aligned} 0 &= \mathcal{M}\{(D_1L(\underline{u}(t),t)+D_2L(\underline{u}(t+r),t+r)).h(t) \\ &\quad +(D_3L(\underline{u}(t),t)+D_4L(\underline{u}(t+r),t+r)).h'(t)\}_t \\ &= \mathcal{M}\{(D_1L(\underline{u}(t),t).h(t)+D_2L(\underline{u}(t),t)).h(t-r) \\ &\quad +(D_3L(\underline{u}(t),t)h'(t)+D_4L(\underline{u}(t),t)).h'(t-r)\}_t \\ &= DJ(u).h. \end{aligned}$$

Since $AP^1(\mathbb{R}^n)$ is dense in $B^{1,2}(\mathbb{R}^n)$, we have $DJ(u).h = 0$ for all $h \in B^{1,2}(\mathbb{R}^n)$, therefore $DJ(u) = 0$. □

Note that the Theorem 4.3 is an extension to the nonautonomous case of Theorem 4 in [8].

Theorem 4.5 (Existence, Uniqueness)**.** *Let $L : (\mathbb{R}^n)^4 \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies* (4.1)(4.2)*. And which also satisfies the two following conditions :*

$$L(.,t) : (\mathbb{R}^n)^4 \to \mathbb{R} \text{ is convex for all } t \in \mathbb{R}. \tag{4.3}$$

$$\begin{cases} \text{There exist } j \in \{1,2\}, k \in \{3,4\} \text{ and } c \in (0,\infty) \\ \text{such that, for all } (x_1,x_2,x_3,x_4,t) \in (\mathbb{R}^n)^4 \times \mathbb{R}, \\ \text{we have : } L(x_1,x_2,x_3,x_4,t) \geq c(|x_j|^2 + |x_k|^2) \end{cases} \tag{4.4}$$

Then there exists a function $u \in B^{1,2}(\mathbb{R}^n)$ which is a weak Besicovitch-ap solution of equation (1.1)*.*
Moreover, if in addition we assume the following condition fulfilled :

$$\begin{cases} \text{There exists } i \in \{1,2\}, l \in \{3,4\} \text{ and } c_1 \in (0,\infty) \\ \text{such that the funtion } M : (\mathbb{R}^n)^4 \times \mathbb{R} \to \mathbb{R}, \text{ defined by} \\ M(x_1,x_2,x_3,x_4,t) := L(x_1,x_2,x_3,x_4,t) - \frac{c_1}{2}|x_i|^2 - \frac{c_1}{2}|x_l|^2, \\ \text{is convex with respect to } (x_1,x_2,x_3,x_4,t) \text{ for all } t \in \mathbb{R}, \end{cases} \tag{4.5}$$

Then the weak Besicovitch-ap solution of (1.1) *is unique.*

Proof. By using Theorem 4.3, the functional J is of class $\mathcal{C}^1$ and, by using (4.3), J is a convex functional. Assumption (4.4) ensures that, for all $u \in B^{1,2}(\mathbb{R}^n)$, we have

$$J(u) \geq c(\mathcal{M}\{|u|^2\} + \mathcal{M}\{|\nabla u|^2\}) = c.\,\|u\|_{1,2}^2 .$$

Since the mean value is translation invariant consequently J is coercive on $B^{1,2}(\mathbb{R}^n)$, and so, [11] p.46, there exists $u \in B^{1,2}(\mathbb{R}^n)$ such that $J(u) = \inf J$. Therefore we have $DJ(u) = 0$ and by using Theorem 4.3, u is a weak Besicovitch-ap solution of (1.1). The existence is proven.

To treat the uniqueness, we note that, under (4.5), the functional $I : B^{1,2}(\mathbb{R}^n) \to \mathbb{R}$, defined by $I(u) := J(u) - \frac{c_1}{2}\mathcal{M}\{|u|^2\} - \frac{c_1}{2}\mathcal{M}\{|\nabla u|^2\}$, is convex and since J is of class $\mathcal{C}^1$, I is also of class $\mathcal{C}^1$. Note that we have $DI(u) = DJ(u) - c_1 \langle u \mid .\rangle$. By using the Minty-monotonicity of the differential of a convex functional, for all $u, v \in B^{1,2}(\mathbb{R}^n)$ we have :

$$0 \leq \langle DI(u) - DI(v), u - v\rangle = \langle DJ(u) - DJ(v), u - v\rangle - c_1 \langle u - v \mid u - v\rangle$$

$$\Rightarrow \langle DJ(u) - DJ(v), u - v\rangle \geq c_1 \|u - v\|_{1,2}^2 .$$

Now if u and v are two weak Besicovitch-ap solutions of (1.1), by using Theorem 4.3 we have $DJ(u) = DJ(v) = 0$, and consequently $c_1 \|u-v\|_{1,2}^2 = 0$, therefore $u = v$. $\square$

Theorem 4.6 (Existence and Density)**.** *Let $K \in \mathcal{C}^2((\mathbb{R}^n)^4, \mathbb{R})$ be a function which satsfies the following conditions :*

$$\text{There exists } a_0 \in [0,\infty) \text{ such that } |K(x)| \leq a_0 |X|^2 \text{ for all } X \in (\mathbb{R}^n)^4. \tag{4.6}$$

$$\begin{cases} \text{There exists } j \in \{1,2\}, k \in \{3,4\} \text{ and } c \in (0,\infty) \\ \text{such that the function } G : (\mathbb{R}^n)^4 \to \mathbb{R}, \text{ defined by} \\ G(x_1,x_2,x_3,x_4) := K(x_1,x_2,x_3,x_4) - \frac{c}{2}|x_j|^2 - \frac{c}{2}|x_k|^2, \\ \text{is convex and non negative on } (\mathbb{R}^n)^4 \end{cases} \tag{4.7}$$

$$\text{The differential } DK \text{ is Lipschitzian on } (\mathbb{R}^n)^4. \tag{4.8}$$

Then the following conclusions hold :

(i) *For all $b \in B^2(\mathbb{R}^n)$ there exixts a unique $u \in B^{1,2}(\mathbb{R}^n)$ which is a weak Besicovitch-ap solution of* (1.2)*.*

(ii) *The set of the $b \in AP^0(\mathbb{R}^n)$ for which there exists a Bohr-ap solution of* (1.2) *is dense in $AP^0(\mathbb{R}^n)$ with respect to the norm*

$$\|b\|_* := \sup\{\mathcal{M}\{b.h\} : h \in B^{1,2}(\mathbb{R}^n), \|h\|_{1,2} \leq 1\}$$

Proof. We introduce the functionals E and E_1 from $B^{1,2}(\mathbb{R}^n)$ in $\mathbb{R}$ setting $E(u) := \mathcal{M}\{K(\underline{u}(t))\}_t$ and $E_1(u) := \mathcal{M}\{G(\underline{u}(t))\}_t$. They are special cases of the functional J of the Theorem 4.3, and consequently they are of class $\mathcal{C}^1$. Note that $E_1(u) = E(u) - \frac{c}{2}\|u\|_{1,2}^2$. By using the F. Riesz isomorphism $j : B^{1,2}(\mathbb{R}^n) \to B^{1,2}(\mathbb{R}^n)^*$, $\langle j(u), v\rangle = \langle u, v\rangle$ for all $u, v \in B^{1,2}(\mathbb{R}^n)$, we can define the gradients $gradE(u) := j^{-1}(DE(u))$ and $gradE_1(u) := j^{-1}(DE_1(u))$. By using the Minty-monotonicity of $gradE_1$ (due to the convexity of E_1) we have, for all $u, v \in B^{1,2}(\mathbb{R}^n)$,

$$\begin{aligned} 0 &\leq \langle gradE_1(u) - gradE_1(v) \mid u - v\rangle \\ &= \langle gradE(u) - gradE(v) \mid u - v\rangle - c.\|u-v\|_{1,2}^2 \end{aligned}$$

that implie that $gradE$ is strongly monotone and consequently, [15] p.100, the following proprety holds

$$gradE \text{ is an homeomorphism from } B^{1,2}(\mathbb{R}^n) \text{ on } B^{1,2}(\mathbb{R}^n) \tag{4.9}$$

We associate to $b \in B^2(\mathbb{R}^n)$ the functional $b^\# \in B^{1,2}(\mathbb{R}^n)^*$ by setting $\langle b^\#, h\rangle := \mathcal{M}\{b(t+r).h(t)\}_t$.
Therefore we have $j^{-1}(b^\#) \in B^{1,2}(\mathbb{R}^n)$ and by using (4.9), there exists $u \in B^{1,2}(\mathbb{R}^n)$ such that $gradE(u) = j^{-1}(b^\#)$, i.e. $DE(u) = b^\#$ which means that, for all $h \in B^{1,2}(\mathbb{R}^n)$,

$$\mathcal{M}\{DK(\underline{u}(t)).\underline{h}(t)\}_t = \mathcal{M}\{b(t+r).h(t)\}_t \quad i.e.$$

$$\begin{aligned} &\mathcal{M}\{[D_1K(\underline{u}(t)) + D_2K(\underline{u}(t+r)) - b(t+r)].h(t) \\ &+ [D_3K(\underline{u}(t)) + D_4K(\underline{u}(t+r)) - b(t+r)].\nabla h(t)\}_t = 0 \end{aligned}$$

and by using Proposition 10 in [8], we obtain that u is a weak Besicovitch-ap solution of (1.2).
About the uniqueness, note that if v is a weak Besicovitch-ap solution of (1.2), then we verify that $\mathcal{M}\{DK(\underline{v}(t)).\underline{h}(t)\}_t = \mathcal{M}\{b(t+r).h(t)\}_t$ for all $h \in B^{1,2}(\mathbb{R}^n)$, and

consequently $DE(v) = b^{\#}$, i.e. $gradE(v) = j^{-1}(b^{\#}) = gradE(u)$, and by using (4.9) we have $u = v$. And so (i) is proven.

Now we introduce the nonlinear unbounded operator

$$\mathcal{K} : Dom(\mathcal{K}) \subset B^{1,2}(\mathbb{R}^n) \to B^2(\mathbb{R}^n)$$

defined by

$$(\mathcal{K}(u))(t) := D_1K(\underline{u}(t-r)) + D_2K(\underline{u}(t)) - \nabla\left[D_3K(\underline{u}(t-r)) + D_4K(\underline{u}(t))\right].$$

And so $\mathcal{K}(u) = b$ means that u is a weak Besicovitch-ap solution of (1.2). By using the assertion (i), $\mathcal{K}$ is bijective. We verify that

$$\|\mathcal{K}(u) - \mathcal{K}(v)\|_* = \|gradE(u) - gradE(v)\|_{1,2}$$

for all $u, v \in Dom(\mathcal{K})$, and by using (4.9) we see that $\mathcal{K}$ is an homeomorphism from $Dom(\mathcal{K})$ on $B^2(\mathbb{R}^n)$. Since $AP^2(\mathbb{R}^n)$ is dense in $B^{1,2}(\mathbb{R}^n)$, $\mathcal{K}(AP^2(\mathbb{R}^n))$ is dense in $B^2(\mathbb{R}^n)$ with respect to the norm $\|.\|_*$, and since $\mathcal{K}(AP^2(\mathbb{R}^n)) \subset AP^0(\mathbb{R}^n) \subset B^2(\mathbb{R}^n)$, we have proven (ii). □

This result is an extension to the neutral delay equations of Theorem 5 in [8].

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